Ministry of Higher Education &
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University of Anbar
College of Education for pure sciences
Phys. Dep.

Calculus Lectures

(First Stage/ Phys. Dep.)

Preparation of Teacher

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Lecture One

Chapter One

Revision and Basic Concepts

1- Intervals

Definition: If a and b are real numbers, we define the intervals as follows:

1- Open intervals $(a,b)=\{x\in\mathbb{R}, a<x<b\}$.

2- Closed intervals $[a,b]=\{x\in\mathbb{R}, a\le x\le b\}$.

3- Half-Open intervals $[a,b)=\{x\in\mathbb{R}, a\le x<b\}$.

4- Half-Closed intervals $(a,b]=\{x\in\mathbb{R}, a<x\le b\}$.

5- $[a,\infty)=\{x\in\mathbb{R}, a\le x<\infty\}$.

6- $(-\infty,b]=\{x\in\mathbb{R}, -\infty<x\le b\}$.

7- $(-\infty,\infty)=\mathbb{R}=\{x\in\mathbb{R}, -\infty<x<\infty\}$.

2- Inequalities

Rules of inequalities

1- If $a-b>0 \quad \leftrightarrow \quad a>b \quad or \quad b<a \quad \forall a,b\in\mathbb{R}$.

2- If $a>b \quad and \quad b>c \quad then \quad a>c \quad \forall a,b,c\in\mathbb{R}$.

3- If $a>b \quad then \quad a\pm c>b\pm c \quad \forall a,b,c\in\mathbb{R}$.

4- If $a>b \quad then \quad \begin{matrix} a\cdot c>b\cdot c & if\ c>0 \\ a\cdot c<b\cdot c & if\ c<0 \end{matrix} \quad \forall a,b,c\in\mathbb{R}$.

Solution set of inequalities

The solution set of an inequality consists of the set real numbers for which the inequality is true state ment if two inequalities have the same solution set, they are said to be equivalent.

Example 1:- Find the solution set of the following inequalities.

1- $3x - 8 < x - 2$

Solve:

$3x - 8 < x - 2 \quad \Rightarrow 3x - 8 + 8 < x - 2 + 8$

$\Rightarrow 3x < x + 6$

$\Rightarrow 3x - x < x - x + 6$

$\Rightarrow 2x < 6$

$\Rightarrow 2x \cdot \frac{1}{2} < 6 \cdot \frac{1}{2}$

$\Rightarrow x < 3$

$\Rightarrow S = \{x \in \mathbb{R}, -\infty < x < 3\} = (-\infty, 3).$

2- $\frac{2x-3}{x+2} < \frac{1}{3} \quad , x \neq -2$

Solve:

If $x + 2 > 0 \quad \Rightarrow 3(2x - 3) < x + 2$

$\Rightarrow 6x - 9 < x + 2$

$$\Rightarrow 5x < 11$$

$$\Rightarrow x < \frac{11}{5}$$

$$\Rightarrow S = \left\{x \in \mathbb{R}, x < \frac{11}{5} \quad and \quad x > -2\right\} = \left(-2, \frac{11}{5}\right).$$

If $x + 2 < 0$ $\quad \Rightarrow 3(2x - 3) > x + 2$

$$\Rightarrow 6x - 9 > x + 2$$

$$\Rightarrow 5x > 11$$

$$\Rightarrow x > \frac{11}{5}$$

$$\Rightarrow S = \left\{x \in \mathbb{R}, x > \frac{11}{5} \quad and \quad x < -2\right\} = \emptyset.$$

3- $x^2 - 3x + 2 < 0.$ H.W.

4- $x(x + 2) \leq 24.$ H.W.

3- Absolute Value

<u>Definition</u>:- The absolute value of real number a is defined as:

$$|a| = \begin{cases} a & if \quad a \geq 0 \\ -a & if \quad a < 0 \end{cases}$$

<u>Some Properties of Absolute Value</u>

1- $|x| < a \quad \Leftrightarrow \quad -a < x < a \qquad \forall a \in \mathbb{R}$

2- $|x| > a \quad \Leftrightarrow \quad x > a \quad or \quad x < -a \qquad \forall a \in \mathbb{R}$

3- $|a+b| \leq |a|+|b| \qquad \forall a,b \in \mathbb{R}$

4- $|a \cdot b| = |a| \cdot |b| \qquad \forall a,b \in \mathbb{R}$

5- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \qquad \forall a,b \in \mathbb{R}$

6- $|a-b| = |b-a| \qquad \forall a,b \in \mathbb{R}$

7- $|a| = \sqrt{a^2}$

Example:- $|3x-2| < 10$

Solve:

$$|3x-2| < 10 \quad \Rightarrow -10 < 3x-2 < 10$$

$$\Rightarrow -8 < 3x < 12$$

$$\Rightarrow -\frac{8}{3} < x < 4$$

$$\Rightarrow S = \left\{x \in \mathbb{R},\ \frac{-8}{3} < x < 4\right\} = \left(\frac{-8}{3}, 4\right).$$

Example:- $|4+2x| \geq x+1$

Solve:

$$|4+2x| \geq x+1 \quad \Rightarrow 4+2x \geq x+1 \quad or \quad 4+2x \leq -(x+1)$$

$$\Rightarrow x \geq -3 \quad or \quad x \leq \frac{-5}{3}$$

$$\Rightarrow S = \{x \in \mathbb{R},\ x \geq -3\} \cup \left\{x \in \mathbb{R},\ x \leq \frac{-5}{3}\right\} = \mathbb{R}.$$

H.W.

1- $2x-3<7$

2- $2x+4<x-4$

3- $\frac{4}{x}<\frac{3}{5}$

4- $\left|\frac{x+3}{6-5x}\right|\leq 2$

5- $\frac{x-2}{x+3}<\frac{x+1}{x}$

6- $|x(x+1)|\leq|x+4|$

Lecture Two

4- The Functions

Definition:- A relation between two set A and B , $f: A\rightarrow B$ is called a function if and only if for each element $x\in A$ their exist unique element $y\in B$ such that $y=f(x)$.

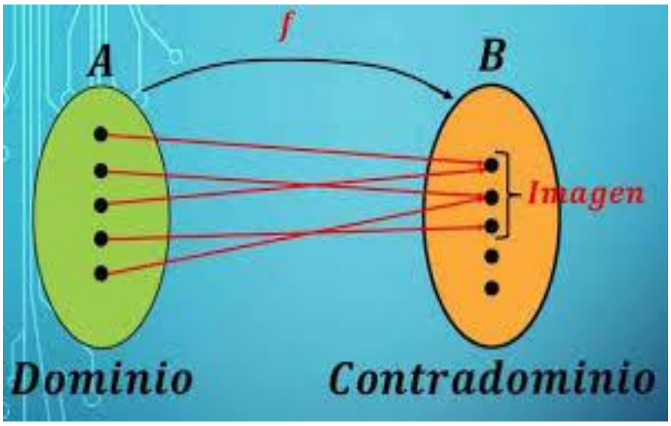
Notes

1- $(x,y)\in f \quad \Longrightarrow \quad y=f(x)$.

2- The set A is called the domain D_f .

3- The set B is called the co-domain.

4- The set of all element of B such that $y=f(x)$ is called the range and represented R_f



Example:- Find the Domain and the Range for each functions

1- $y = x^2$ $\Longrightarrow$ **Domain = R , Range = R**

2- $y = x + 3$ $\Longrightarrow$ **Domain = R , Range = R**

3- $y = \sqrt{x - 4}$

$$\Longrightarrow x - 4 \geq 0 \Longrightarrow x - 4 + 4 \geq 0 + 4 \Longrightarrow x \geq 4$$

Then $D_f = \{x: x \geq 4\}, \quad R_f = \{y: y \geq 0\}$

4- $y = \frac{x-3}{x+2}$

$$\Longrightarrow x + 2 = 0 \Longrightarrow x = -2 \Longrightarrow D_f = R/\{-2\}$$

$$\Longrightarrow y(x + 2) = x - 3 \Longrightarrow yx + 2y = x - 3$$

$$\Longrightarrow yx - x = -3 - 2y \Longrightarrow x(y - 1) = -3 - 2y$$

$$\Longrightarrow x = \frac{-3 - 2y}{y - 1}$$

$$\Longrightarrow y - 1 = 0 \Longrightarrow y = 1 \Longrightarrow R_f = R/\{1\}$$

H.W.

Find the Domain and the Range for the functions

1- $y = \frac{1}{x-2}$

2- $f(x) = \frac{1}{\sqrt{x+3}}$

3- $y = x^2 - 5x + 6$

4- $y = \sqrt{x^2 - 9}$

5- $y = \sqrt{x^2 - 2x - 3}$

6- $f(x) = \frac{\sqrt{x-1}}{x^2+4}$

Some types of Function

Definition 1:- Absolute Value Function is define by

$$f(x) = |x| = \begin{Bmatrix} x & if & x \geq 0 \\ -x & if & x < 0 \end{Bmatrix}.$$

Definition 2:- A function is called even function if $f(-x) = f(x)$.

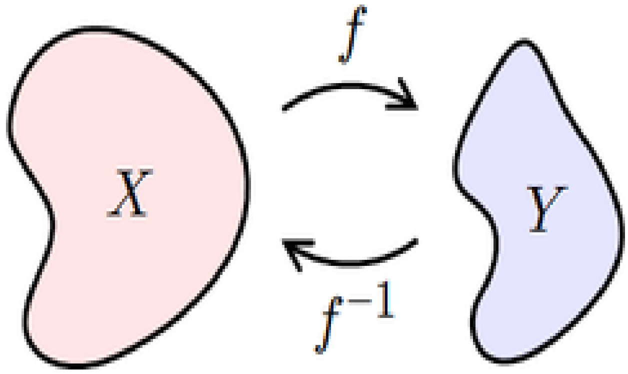
Definition 3:- A function is called odd function if $f(-x) = -f(x) \neq f(x)$.

Definition 4:- A function is called constant function if $f(x) = a_0$, $\forall a \in \mathbb{R}$.

Definition 5:- A function is called linear function if $f(x) = a_1 x + a_0$,$\forall a \in \mathbb{R}$.

Definition 6:- A function subjective $f(x): X \to Y$, we define the invers function such that $x = f^{-1}(y): Y \to X$.

$$f\left(f^{-1}(y)\right) = x \qquad\qquad D_{f^{-1}} = R_f, D_f = R_{f^{-1}}$$



Example:- $y = f(x) = x^3$ **Find invers function and** $D_{f^{-1}}, R_{f^{-1}}$

Solve:

$$y = x^3 \Rightarrow x = \sqrt[3]{y} = f^{-1}(y)$$

$$D_{f^{-1}} = \mathbb{R}^+, \qquad R_{f^{-1}} = \mathbb{R}.$$

Composite of Function

Definition:- **If we have the two functions** $f(x), g(x)$ **then we define a composite function as**

$$z = f\left(g(x)\right) = f \circ g(x) \quad or \quad z = g\left(f(x)\right) = g \circ f(x)$$

$$X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad g \quad} Z$$

Example:- $f(x) = \sqrt{x},\ \ g(x) = x^2$ **Find** $f \circ g(x) \quad and \quad g \circ f(x).$

Solve:

$$f \circ g(x) = f\big(g(x)\big) = f(x^2) = \sqrt{x^2} = x$$

$$g \circ f(x) = g\big(f(x)\big) = g\big(\sqrt{x}\big) = x$$

Example:- $f(x) = x^3 \quad g(x) = 2x$ **Find** $f \circ g(x) \quad and \quad g \circ f(x)\ with \quad x = 2.$

Solve:

$$f \circ g(x) = f\big(g(x)\big) = f(2x) = (2x)^3 = 8x^3 = 64$$

$$g \circ f(x) = g\big(f(x)\big) = g(x^3) = 2x^3 = 16$$

H.W.

Find $f \circ g(x) \quad and \quad g \circ f(x)$

1- $f(x) = x + 1 \quad g(x) = x^2.$

2- $f(x) = x^2 - 6x + 2 \quad g(x) = -2x.$

3- $f(x) = 2x^2 + 3 \quad g(x) = 4x^3 + 1\ , \quad with \quad x = 1.$

Lecture Three

5- Properties of Exponential

For all numbers a, b **the following rules are satisfies :**

1- $e^a \cdot e^b = e^{a+b}$

2- $\frac{e^a}{e^b} = e^{a-b}$

3- $e^{-a} = \frac{1}{e^a}$

4- $(e^a)^k = e^{ak}$

5- $e^0 = 1$

6- $e^{-\infty} = 0$

6- Properties of Natural Logarithm $\ln(x)$.

For any $a, b > 0$, then the following rules are satisfies :

1- $\ln(ab) = \ln(a) + \ln(b)$

2- $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

3- $\ln(a)^k = k\ln(a)$

4- $\ln(1) = 0$

5- $\ln e^x = x$

6- $e^{\ln x} = x$

7- The Equation of a Straight line

1-<u>Find the Slope of a Straight line</u>

- Given a line (L) passing through the point (x_1, y_1) and (x_2, y_2) if (m) is the slope then

$$m = \tan(\theta) = \frac{\Delta y}{\Delta x} \Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Given an equation line (L) $ax + by + c = 0$ then the slope

$$m = \frac{-a}{b}$$

Note:

If m_1 and m_2 are slopes we said to be the two lines parallel if $m_1 = m_2$, and said to be the two lines orthogonal if $m_1 \times m_2 = -1$.

2- Find the equation of a Straight line

- Equation of a straight line where slope $= m$ and passing through the point $P(x_1, y_1)$

$$y - y_1 = m(x - x_1)$$

- Equation of a straight line passing through points $(x_1, y_1), (x_2, y_2)$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

قوانين النســـب المثلثية في الأرباع

الربع الأول

1) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

2) $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

3) $\tan\left(\frac{\pi}{2} - x\right) = \cot x$

4) $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

الربع الثاني

1) $\sin(\pi - x) = \sin x$

2) $\cos(\pi - x) = -\cos x$

3) $\tan(\pi - x) = -\tan x$

4) $\cot(\pi - x) = -\cot x$

الربع الثالث

1) $\sin(\pi + x) = -\sin x$

2) $\cos(\pi + x) = -\cos x$

3) $\tan(\pi + x) = \tan x$

4) $\cot(\pi + x) = \cot x$

الربع الرابع

1) $\sin(-x) = -\sin x$

2) $\cos(-x) = \cos x$

3) $\tan(-x) = -\tan x$

4) $\cot(-x) = -\cot x$

قوانيـــن القـــــوى

1) $\frac{x^n}{x^m} = x^{n-m}$

2) $x^n * x^m = x^{n+m}$

3) $(x^n)^m = x^{\mathrm{nm}}$

4) $(\sqrt[n]{x})^m = x^{\frac{m}{n}}$

قوانيـــن اللوغاريتـــمات

1) $\log_a 1 = 0$

2) $\log_a a = 1$

3) $\log_a b^m = m \log_a b$

4) $\log_a a^m = m$

5) $\log_a(b * c) = \log_a b + \log_a c$

6) $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$

7) $\log_a\left(\frac{1}{b}\right) = -\log_a b$

8) $\log_{10} a = \ln a$

9) $e^{e^{\ln x}} = x$

النسب المثلثيه لمجموع وفرق زاويتين

1) $\sin(a + b) = \sin a \cos b + \cos a \sin b$

2) $\sin(a - b) = \sin a \cos b - \cos a \sin b$

3) $\cos(a + b) = \cos a \cos b - \sin a \sin b$

4) $\cos(a - b) = \cos a \cos b + \sin a \sin b$

5) $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$

6) $\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

قوانيـن النســـب المثلثيـــة

1- $\cos^2 x + \sin^2 x = 1$

2- $1 + \tan^2 x = \sec x$

3- $1 + \cot^2 x = \csc x$

4- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

5- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

تحويل مجموع وفرق جيبي وجيبي تمام إلى حاصل ضرب

1) $\sin a+\sin b=2\sin\frac{A+B}{2}\cos\frac{a-b}{2}$

2) $\sin a-\sin b=2\cos\frac{a+b}{2}\sin\frac{a-b}{2}$

3) $\cos a+\cos b=2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$

4) $\cos a-\cos b=-2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$

قوانين المفكـــــــــــــــــوك

1) $(1+x)^n=1+\frac{nx}{1!}+\frac{n(n-1)x^2}{2!}+\cdots$

2) $(a^2-b^2)=(a-b)(a+b)$

3) $(a^2+b^2)=(a+i)(a-i)$

4) $(a^3-b^3)=(a-b)(a^2+ab+b^2)$

5) $(a^3+b^3)=(a+b)(a^2-ab+b^2$

النسب المثلثية لمضاعف الزاوية

1) $\sin 2a=2\sin a\cos a$

ومنه

$\sin a=2\sin\frac{a}{2}\cos\frac{a}{2}$

2) $\cos 2a=\cos^2 a-\sin^2 a$

ومنه

$\cos 2a=1-2\sin^2 a$

$\cos 2a=2\cos^2 a-1$

3- $\tan 2a=\frac{2\tan a}{1-\tan^2 a}$

النسب المثلثية لنصف الزاوية

1) $\sin\frac{a}{2}=\pm\sqrt{\frac{1-\cos a}{2}}$

2) $\cos\frac{a}{2}=\pm\sqrt{\frac{1+\cos a}{2}}$

3) $\tan\frac{a}{2}=\pm\sqrt{\frac{1-\cos a}{1+\cos a}}$

Lecture Four

Chapter Two

Theorem of Limit

1- Not the following Rules hold if

$$\lim_{x\to a} f(x) = L \quad and \qquad \lim_{x\to a} g(x) = M$$

8- $\lim_{x\to a} c = c \quad , \quad where \quad c \in \mathbb{R}.$

9- $\lim_{x\to a} f(x)c = c \lim_{x\to a} f(x) = cL \quad , \quad where \quad c \in \mathbb{R}.$

10- $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M\,.$

11- $\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M\,.$

12- $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, \qquad where\; M \neq 0.$

13- $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n \quad , \quad where \quad n \in \mathbb{N}.$

Example:- Evaluate the following Limits.

$$1 - \lim_{x\to 0}\left[\frac{x^4 - x + 1}{x - 1}\right]^3$$

Solve:

$$\lim_{x\to 0}\left[\frac{x^4 - x + 1}{x - 1}\right]^3 = \left[\lim_{x\to 0}\frac{x^4 - x + 1}{x - 1}\right]^3$$

$$= \left[\frac{\lim\limits_{x\to 0} x^4 - \lim\limits_{x\to 0} x + \lim\limits_{x\to 0} 1}{\lim\limits_{x\to 0} x - \lim\limits_{x\to 0} 1}\right]^3 = \left[\frac{0 - 0 + 1}{0 - 1}\right]^3 = -1.$$

$$2-\lim_{x\to 5}\left[\frac{x^2-25}{x+5}\right]\left[\frac{x^2-25}{x-5}\right]$$

Solve:

$$\lim_{x\to 5}\left[\frac{x^2-25}{x+5}\right]\left[\frac{x^2-25}{x-5}\right]=\lim_{x\to 5}\left[\frac{x^2-25}{x+5}\right]\lim_{x\to 5}\left[\frac{x^2-25}{x-5}\right]$$

$$=\left[\frac{25-25}{5+5}\right]\lim_{x\to 5}\left[\frac{(x-5)(x+5)}{x-5}\right]$$

$$=\frac{25-25}{5+5}(5+5)=0.$$

$$3-\lim_{y\to 2}\frac{\sqrt{y^2+12}-4}{y-2}$$

Solve:

$$\lim_{y\to 2}\frac{\sqrt{y^2+12}-4}{y-2}=\lim_{y\to 2}\frac{\left(\sqrt{y^2+12}-4\right)\left(\sqrt{y^2+12}+4\right)}{(y-2)\left(\sqrt{y^2+12}+4\right)}$$

$$=\lim_{y\to 2}\frac{y^2+12-16}{(y-2)\left(\sqrt{y^2+12}+4\right)}=\lim_{y\to 2}\frac{y^2-4}{(y-2)\left(\sqrt{y^2+12}+4\right)}$$

$$=\lim_{y\to 2}\frac{(y-2)(y+2)}{(y-2)\left(\sqrt{y^2+12}+4\right)}=\lim_{y\to 2}\frac{y+2}{\left(\sqrt{y^2+12}+4\right)}$$

$$\frac{4}{4+4}=\frac{1}{2}$$

$$4-\lim_{t\to 4}\frac{t-4}{t^2-t-12} \qquad H.W.$$

$$5-\lim_{x\to -1}\frac{x^3+x+2}{x+1} \qquad H.W.$$

Example:- If $f(x)=x^2-x$ then find $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Solve:

Since $f(x)=x^2-x,\quad f(x+h)=(x+h)^2-(x+h) \qquad and$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-(x+h)-(x^2-x)}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2hx+h^2-x-h-x^2+x}{h}=\lim_{h\to 0}\frac{2hx+h^2-h}{h}$$

$$=\lim_{h\to 0}\frac{h(2x+h-1)}{h}=\lim_{h\to 0}(2x+h-1)$$

$$2x+0-1=2x-1$$

2- Infinite Limits

Some times we need to know what happens to $f(x)$ as x gets large and positive $(x\to\infty)$ or large and negative $(x\to-\infty)$ consider a function $f(x)=\frac{1}{x}$ what dose $\lim_{x\to\infty} f(x)$,

$f(x)$ gets close to 0, as x gets large and large , this is written

$$\lim_{x\to\infty}\frac{1}{x}=0 \qquad or \qquad \lim_{x\to-\infty}\frac{1}{x}=0$$

Example:- Find the following Limits if they exist

$$1-\lim_{x\to\infty}\frac{x^3+2x+1}{3x^3+1}$$

Solve:

$$\lim_{x\to\infty}\frac{x^3+2x+1}{3x^3+1}=\lim_{x\to\infty}\frac{\frac{x^3}{x^3}+\frac{2x}{x^3}+\frac{1}{x^3}}{\frac{3x^3}{x^3}+\frac{1}{x^3}}$$

$$=\lim_{x\to\infty}\frac{1+\frac{2}{x^2}+\frac{1}{x^3}}{3+\frac{1}{x^3}}=\frac{1}{3}$$

$$2-\lim_{x\to\infty}\frac{4x-2}{x^2+3}$$

Solve:

$$\lim_{x\to\infty}\frac{4x-2}{x^2+3}=\lim_{x\to\infty}\frac{\frac{4x}{x^2}-\frac{2}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}}$$

$$=\lim_{x\to\infty}\frac{\frac{4}{x}-\frac{2}{x^2}}{1+\frac{3}{x^2}}=\frac{0}{1}=0$$

$$3-\lim_{x\to\infty}(x-\sqrt{x^2+x})$$

Solve:

$$\lim_{x\to\infty}(x-\sqrt{x^2+x})=\lim_{x\to\infty}(x-\sqrt{x^2+x})\frac{(x+\sqrt{x^2+x})}{(x+\sqrt{x^2+x})}$$

$$= \lim_{x\to\infty} \frac{(x^2 - (x^2 + x))}{(x + \sqrt{x^2 + x})} = \lim_{x\to\infty} \frac{-x}{(x + \sqrt{x^2 + x})}$$

$$= \lim_{x\to\infty} \frac{-x}{(x + \sqrt{x^2 + x})} = \lim_{x\to\infty} \frac{-\frac{x}{x}}{(\frac{x}{x} + \sqrt{\frac{x^2}{x^2} + \frac{x}{x^2}})}$$

$$= \lim_{x\to\infty} \frac{-1}{(1 + \sqrt{1 + \frac{1}{x}})} = \frac{-1}{(1 + \sqrt{1 + 0})} = \frac{-1}{2}$$

$4 - \lim_{x\to\infty} \sqrt{\frac{9x - 1}{x + 1}}$ $\quad H.W.$

$5 - \lim_{x\to\infty} \frac{\sqrt{x^2 + x}}{x + 1}$ $\quad H.W.$

Notes

$1 - \lim_{x\to 0} \frac{\sin x}{x} = 1$

$2 - \lim_{x\to 0} \frac{1 - \cos x}{x} = 0$

$3 - \lim_{x\to 0} \frac{\tan x}{x} = 1$

$4 - \lim_{x\to 0} \frac{\sin ax}{ax} = 1$

Lecture Five

3- Right and Left Limit

Example:- Is $\lim_{x\to 0}\frac{|x|}{x}$ exist at $x=0$?

Solve:

$$1-\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=\lim_{x\to 0^+}(1)=1$$

$$2-\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$$

$$\lim_{x\to 0^+}\frac{|x|}{x}\neq\lim_{x\to 0^-}\frac{|x|}{x}$$

Then Limit is not exist at $x=0$

Example:-

$$f(x)=\begin{cases}2x+1 & x>1\\ 5 & x=1\\ 7x^2-4 & x<1\end{cases}$$

Solve:

$$1-\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}(2x+1)=\lim_{x\to 1^+}(2\cdot 1+1)=3$$

$$2-\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}(7x^2-4)=\lim_{x\to 1^-}(7\cdot 1-4)=3$$

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^-}f(x)$$

Then Limit is exist at $x=1$, and equal 3.

4- Hopital Rule

Using the Limit Hopital Rule for Ralition function at $\frac{\infty}{\infty}$ or $\frac{o}{o}$ such that derivative

Example:- Find $\lim_{x\to2}\frac{x^2-4}{x-2}$

Solve:

$\lim_{x\to2}\frac{x^2-4}{x-2}=\frac{o}{o}$ **then**

$$\lim_{x\to2}\frac{x^2-4}{x-2}=\lim_{x\to2}\frac{2x}{1}=2\cdot2=4$$

Example:- Find $\lim_{x\to0}\frac{\sin x-x}{x^2}$

Solve:

$\lim_{x\to0}\frac{\sin x-x}{x^2}=\frac{o}{o}$ **then**

$\lim_{x\to2}\frac{\sin x-x}{x^2}=\lim_{x\to2}\frac{\cos x-1}{2x}=\frac{0}{0}$ **then**

$$\lim_{x\to2}\frac{\cos x-1}{2x}=\lim_{x\to2}\frac{-\sin x}{2}=\frac{0}{2}=0$$

Example:- Find the following Limits by using Limit Hopital Rule

$$1-\lim_{x\to0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$$

Solve:

$\lim_{x\to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} = \frac{o}{o}$, then

$\lim_{x\to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{\left(\frac{1}{2}\right)(1+x)^{\frac{-1}{2}}-\frac{1}{2}}{2x} = \frac{0}{0}$, then

$$\lim_{x\to 0} \frac{\left(\frac{1}{2}\right)(1+x)^{\frac{-1}{2}}-\frac{1}{2}}{2x} = \lim_{x\to 0} \frac{-\left(\frac{1}{4}\right)(1+x)^{\frac{-3}{2}}}{2} = \frac{-1}{8}$$

$2 - \lim\limits_{x\to 0} \frac{x-\sin x}{x^3}$

Solve:

$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{o}{o}$, then

$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \frac{0}{0}$, then

$\lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \frac{0}{0}$,then

$$\lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

$3 - \lim\limits_{x\to\frac{\pi}{2}} [\sec x \cdot \tan x]$ $\quad H.W.$

$4 - \lim\limits_{x\to 0} \frac{\ln(x+1)-2x}{x^2}$ $\quad H.W.$

5- Continuity

Definition: We said to be the functions Continuity at x_0 if and only if satisfies condition.

8- $f(x_0)$ is know

9- $\lim_{x\to x_0} f(x)$ is exist

10- $f(x_0) = \lim_{x\to x_0} f(x)$

Example:- Is $f(x) = \begin{cases} \frac{x^3+x}{x} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$ Continuity at $x = 0$?

Solve:

1- $f(0) = 1$

2- $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{x^3+x}{x}\right) = \lim_{x\to 0}(x^2+1) = 1$

3- $f(0) = \lim_{x\to 0} f(x)$

Then the function is Continuity at $x = 0$.

Example:- Is $f(x) = \begin{cases} x & if\ x \leq 0 \\ x+2 & if\ x > 0 \end{cases}$ Continuity at $x = 0$?

Solve:

1- $f(0) = 0$

2- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+2) = 2$ and

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x) = 0$

Then the function is not Continuity at $x = 0$,since Limits is not exist at $x = 0$.

H.W.

Example:- Show that $f(x) = \begin{cases} -x^2 & if\ x < -2 \\ 2x & if\ x \geq -2 \end{cases}$ Continuity at $x = -2$

Example:- Show that $f(x) = \begin{cases} x^3 & if\ x \geq -1 \\ 1 - 2x & if\ x < -1 \end{cases}$ Continuity at $x = -1$

Example:- Find a,b such that the function is continuity at $x = 2$

$$f(x) = \begin{cases} x^3 - ax + b & if\ x > 2 \\ 3 & if\ x = 2 \\ a\sqrt{x+2} + b & if\ x < 2 \end{cases}$$

Solve:

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^3 - ax + b) = 8 - 2a + b$ and

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \left(a\sqrt{x+2} + b\right) = 2a + b$$

Since the function $f(x)$ is continuity, then

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 8 - 2a + b = 2a + b$$

$$\Rightarrow 4a = 8$$

$$\Rightarrow a = 2$$

and

$$8 - 2a + b = 3$$

$$\Rightarrow b = 3 - 8 + 2a$$

$$\Rightarrow b = -5 + 4 = -1$$

Hence $a = 2\ , b = -1$

Lecture Six

Chapter Three

Derivatives

1- Derivative using the definition

The Derivative of the function $y = f(x)$ is the function $y' = f'(x)$ whose value at each x is define by rule

$$y = f(x) \Rightarrow y' = f'(x)$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

<u>Definition:</u> If $y = f(x)$ is a continuous function, then we define the derivative of function as a limit as

$$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Example:- Find the derivative of the function $y = x^2$ by define.

Solve:

$y = f(x) = x^2$ **and**

$f(x+\Delta x) = (x+\Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$ **, then**

$$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[2x + \Delta x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x$$

$$= 2x$$

2- The Rules for Derivative

1- If $y = a \implies \frac{dy}{dx} = 0$, where a is constant.

Example: $y = 2 \implies \frac{dy}{dx} = 0$.

2- If $y = x^n \implies \frac{dy}{dx} = nx^{n-1}$, where n any number.

Example: $y = x^{-2} \implies \frac{dy}{dx} = -2x^{-2-1} = -2x^{-3}$.

3- If $y = ax^n \implies \frac{dy}{dx} = a \cdot nx^{n-1}$.

Example: $y = 4\sqrt[3]{x} \implies \frac{dy}{dx} = 4 \cdot \frac{1}{3} x^{\frac{1}{3}-1} = \frac{4}{3\sqrt[3]{x^2}}$.

4- If $y = u(x) + v(x) \implies \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Example: $y = 2x^2 + 8 - 5x^4 \implies \frac{dy}{dx} = 4x + 0 - 20x^3 = 4x - 20x^3$.

5- If $y = b[u(x)]^n \implies \frac{dy}{dx} = b \cdot n[u(x)]^{n-1} \cdot \frac{du}{dx}$ where b is constant.

Example: $y = 3(2x^2 - x + 4)^7 \implies \frac{dy}{dx} = 3 \cdot 7(x^2 - x + 4)^6 \cdot (4x - 1)$

6- If $y = u(x) \cdot v(x) \Rightarrow \frac{dy}{dx} = u(x) \cdot \frac{dv}{dx} + v(x) \cdot \frac{du}{dx}$

Example: $y = (x^2+1)(x-3)^2 \Rightarrow (x^2+1)[2(x-3)] + (x-3)^2[2x]$

7- If $y = \frac{u(x)}{v(x)} \Rightarrow \frac{dy}{dx} = \frac{v(x)\cdot\frac{du}{dx} - u(x)\cdot\frac{dv}{dx}}{[v(x)]^2}$.

Example: $y = \frac{x^2+1}{3x^2+2x} \Rightarrow \frac{dy}{dx} = \frac{(3x^2+2x)\cdot(2x) - (x^2+1)\cdot(6x+2)}{[3x^2+2x]^2}$

$$= \frac{(6x^3+4x^2) - (6x^3+2x^2+6x+2)}{9x^4+12x^3+4x^2} = \frac{2x^2-6x-2}{9x^4+12x^3+4x^2}$$

3- The Derivative of trigonometric functions

1- $y = \sin(g(x)) \Rightarrow y' = \cos(g(x)) \cdot g'(x)$

2- $y = \cos(g(x)) \Rightarrow y' = -\sin(g(x)) \cdot g'(x)$

3- $y = \tan(g(x)) \Rightarrow y' = \sec^2(g(x)) \cdot g'(x)$

4- $y = \cot(g(x)) \Rightarrow y' = -\csc^2(g(x)) \cdot g'(x)$

5- $y = \sec(g(x)) \Rightarrow y' = \sec(g(x))\tan(g(x)) \cdot g'(x)$

6- $y = \csc(g(x)) \Rightarrow y' = -\csc(g(x))\cot(g(x)) \cdot g'(x)$

Example:- Using the definition of the derivative of a function to find the derivative of the functions

1- $f'(x) = x^3 + 2x$

Solve:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 + 2(x+\Delta x) - (x^3+2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2 + 2$$

$$= 3x^2 + 2$$

2- $y = \sqrt{x}$

Solve:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{(x + \Delta x)} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{(x + \Delta x)} - \sqrt{x}\right)\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)}{\Delta x\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

Example:- Find the derivatives of the following functions.

1- $f(x) = x^7 - x^{-5} + x^3 - 19 \quad \Rightarrow \quad f'(x) = 7x^6 + 5x^{-6} + 3x^2$

2- $g(x) = x\sqrt{x^2 - 1} \quad \Rightarrow \quad g'(x) = \frac{x \cdot 2x}{2\sqrt{x^2-1}} + \sqrt{x^2 - 1} = \frac{2x^2-1}{\sqrt{x^2-1}}$

3- $y = x + \frac{1}{x^2} \quad \Rightarrow \quad y' = 1 + \frac{-2}{x^3} = 1 - \frac{2}{x^3}$

4- The Derivative of Natural Logarithm functions

If y is function given by $y = \ln(g(x))$, where $g(x) > 0$, then the derivative of y is

$$y = \ln(g(x)) \quad \Rightarrow \quad y' = \frac{g'(x)}{g(x)}$$

For Example:-

$1 - y = \ln(x) \quad \Rightarrow \quad y' = \frac{1}{x}$

$2 - y = \ln(x^2 + 2x) \quad \Rightarrow \quad y' = \frac{2x + 2}{x^2 + 2x}$

Lecture Seven

5- The Derivative of Exponential functions

The function $e^{g(x)}$ has the derivative given by

$$y = e^{g(x)} \quad \Rightarrow \quad y' = e^{g(x)} \cdot g'(x)$$

For Example:-

$1 - y = e^{x^2 - x} \quad \Rightarrow \quad y' = e^{x^2 - x} \cdot (2x - 1)$

Example:- Find the derivatives of the following functions.

1- $f(x) = \sin x^2 + \cot(x^4 - 1)$

Solve:

$$f'(x) = 2x\cos x^2 - 4x^3\csc^2(x^4 - 1)$$

2- $g(x) = \sqrt{\csc(x^2) - 1}$

Solve:

$$g'(x) = \frac{-2x\csc x^2 \cot x^2}{2\sqrt{\csc(x^2) - 1}}$$

$$= \frac{-x\csc x^2 \cot x^2}{\sqrt{\csc(x^2) - 1}}$$

3- $y = \ln(2x - x^{-2})$

Solve:

$$y' = \frac{2 + 2x^{-3}}{2x - x^{-2}}$$

4- $f(x) = e^{\frac{1}{x}}$

Solve:

$$\frac{df}{dx} = e^{\frac{1}{x}} \cdot \frac{-1}{x^2}$$

5- $y = \cos(e^{2x})$

Solve:

$$\frac{dy}{dx} = -2e^{2x}\sin(e^{2x})$$

6- $y = (\sec(2x) + \tan(3x))^{-2}$ **H.W.**

7- $g(x) = \ln\sqrt{\frac{1+x}{1-x}}$ **H.W.**

8- $h(x) = x\ln(e^{\cot x})$ **H.W.**

6- The Derivative of $y = a^{g(x)}$, where $a > 0$

If $f(x)$ is a function given in the form $y = f(x) = a^{g(x)}$, then the easiest way to find the derivative y' is by taking logarithms.

$$\ln y = \ln a^{g(x)}$$

$$\Rightarrow \ln y = g(x)\ln a \quad \text{where} \quad \ln a^{g(x)} = g(x)\ln a$$

$$\Rightarrow \frac{y'}{y} = g'(x)\ln a$$

$$\Rightarrow y' = y \cdot g'(x)\ln a$$

$$\Rightarrow y' = a^{g(x)} \cdot g'(x)\ln a \text{ , where } y = a^{g(x)}$$

Thus, if $y = a^{g(x)} \Rightarrow y' = a^{g(x)} \cdot g'(x)\ln a$.

Example:- Find the derivatives of the following functions.

$$1 - f(x) = 2^{(x^4 - x)} \quad \Rightarrow \quad f'(x) = 2^{(x^4-x)}\,(4x^3 - 1)\ln 2$$

$$2 - y = 7^{(\sin x^2)} \quad \Rightarrow \quad y' = 7^{(\sin x^2)}\, 2x\cos x^2 \ln 7$$

$$3 - g(x) = \left(\frac{3}{2}\right)^{\sqrt{x-1}} \quad \Rightarrow \quad g'(x) = \left(\frac{3}{2}\right)^{\sqrt{x-1}} \frac{1}{2\sqrt{x-1}} \ln\left(\frac{3}{2}\right)$$

7- The Derivative of trigonometric reverse functions

$$1 - \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$$2 - \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}$$

$$3 - \frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}$$

$$4 - \frac{d}{dx}\cot^{-1} x = \frac{-1}{1 + x^2}$$

$$5 - \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$$

$$6 - \frac{d}{dx}\csc^{-1} x = \frac{-1}{x\sqrt{x^2 - 1}}$$

Example:- Find $\frac{dy}{dx}$

1- If $y = \sin^{-1}(2x^2)$ $\quad \Rightarrow \quad$ $\frac{dy}{dx} = \frac{4x}{\sqrt{1-(2x^2)^2}}$

2- If $y = \tan^{-1}(x^2 + 2x)$ $\Longrightarrow$ $\frac{dy}{dx} = \frac{2x+2}{1+(x^2+2x)^2}$

3- If $y = \sin^{-1}(x^2 + 3x - \cos(x))$ $\Longrightarrow$ $\frac{dy}{dx} = \frac{2x+3-(-\sin(x))}{\sqrt{1-(x^2+3x-\cos(x))^2}}$

4- If $y = \cos^{-1}(x^2 + \tan(2x))$ H.W.

Example:- If $y = \sin^{-1}(x)$, Prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Solve:

$$y = \sin^{-1}(x) \Longrightarrow x = \sin(y)$$

$$\Longrightarrow 1 = \cos(y)\frac{dy}{dx}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2(y)}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Example:- If $y = \cos^{-1}(x)$, Prove that $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$ H.W.

Lecture Eight

8- The Derivative of Composite functions (Chain Rule)

If y is differentiable function of (u) and (u) is differentiable function of (x).

Then y is a differentiable function of (x).

That is

$$y = f(u) \Rightarrow \frac{dy}{du} \quad , \quad u = f(x) \Rightarrow \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example:- Find $\frac{dy}{dt}$, where $y = x^2 + \sqrt{x}$ and $x = 3t^2 - 2t + 1$

Solve:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \left(2x + \frac{1}{2\sqrt{x}}\right)(6t - 2)$$

Substitute

$$x = 3t^2 - 2t + 1$$

$$\Rightarrow \frac{dy}{dt} = \left(2(3t^2 - 2t + 1) + \frac{1}{2\sqrt{3t^2 - 2t + 1}}\right)(6t - 2)$$

Example:- Find $\frac{dy}{dx}$, where $x = 2t + 3$ and $y = t^2 - 1$

Solve:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2t}{2} = t$$

Substitute

$$t = \frac{x - 3}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x - 3}{2}$$

9- Implicit Derivative

Example:- Find $\frac{dy}{dx}$, if $y^2 = x$

Solve:

$$y^2 = x$$

$$\Rightarrow 2y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$$y^2 = x \quad \Rightarrow \quad y = \pm\sqrt{x} \quad \Rightarrow \quad y_1 = +\sqrt{x} \quad and \quad y_2 = -\sqrt{x}$$

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad and \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = \frac{-1}{2\sqrt{2}}$$

Example:- If $y = f(x)$ for $y^3 + xy + x^2 = 2$. Find $\frac{dy}{dx}$

Solve:

$$\Rightarrow 3y^2\frac{dy}{dx} + x\frac{dy}{dx} + y + 2x = 0$$

$$\Rightarrow 3y^2\frac{dy}{dx} + x\frac{dy}{dx} = -2x - y$$

$$\Rightarrow (3y^2 + x)\frac{dy}{dx} = -2x - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x - y}{3y^2 + x}$$

10-Higher Order Derivative

<u>**Definition:**</u> If $y = f(x)$ is a continuous function, then the first derivative is $y' = \frac{dy}{dx} = f'(x)$ and the second order derivative is $y'' = \frac{d^2y}{dx^2} = f''(x)$. the n^th^ order derivative is

$$y^n = \frac{d^ny}{dx^n} = f^n(x)$$

Example:- If $y = 4x^3 + 2x + 1$, Find $\frac{d^2y}{dx^2}$

Solve:

$$\Rightarrow \quad y' = \frac{dy}{dx} = 12x^2 + 2$$

$$\Rightarrow \; y'' = \frac{d^2y}{dx^2} = 24x$$

11- Derivative with Physical Application

1- Velocity $\Rightarrow$ denoted by v

2- Acceleration $\Rightarrow$ denoted by a

3- Time $\Rightarrow$ denoted by t

4- Distance $\Rightarrow$ denoted by x

$$v = \frac{dx}{dt} \qquad , \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Example:- Find Velocity and Acceleration such that $x = t^3 + 3t^2 + t + 1$, at $t = 2$.

Solve:

$$\Rightarrow v = \frac{dx}{dt} = 3t^2 + 6t + 1$$

$$\Rightarrow v = 12 + 12 + 1 = 25m/s \quad \text{at } t = 2$$

$$\Rightarrow a = \frac{dv}{dt} = 6t + 6$$

$$\Rightarrow a = 12 + 6 = 18m/s \quad \text{at } t = 2$$

Lecture Nine

Chapter Four

Integration

If $F(x)$ is a function whose derivative $F'(X) = f(x)$, then $F(x)$ is called an integration of $f(x)$, and we will write as.

$$\int f(x)dx = F(x) + c \quad \text{where } c \text{ any constant}$$

Note that the integration can be used to find Area, Volume, Velocity, …

1- Properties of Integrals

Let $f(x)$ and $g(x)$ be integrable. Then ,

$$1 - \int c\, f(x)dx = c \int f(x)dx\,, \quad \text{where } c \text{ constant}$$

$$2 - \int (f(x)dx \pm g(x)dx) = \int f(x)dx \pm \int g(x)dx$$

$$3 - \int u^n du = \frac{u^{n+1}}{n+1} + c\,, \quad \text{where } n \neq -1$$

$$4 - \int \frac{du}{u} = \ln|u| + c$$

$$5 - \int a^u du = \frac{a^u}{\ln(a)} + c\,, \quad \text{where} \quad a > 0$$

$$6 - \int e^u du = e^u + c$$

$$7-\int \sin(x)dx = -\cos(x) + c$$

$$8-\int \cos(x)\, dx = \sin(x) + c$$

$$9-\int \tan(x)\, dx = \ln|\sec(x)| + c$$

$$10-\int \cot(x)\, dx = \ln|\sin(x)| + c$$

$$11-\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + c$$

$$12-\int \csc(x)dx = \ln|\csc(x) - \cot(x)| + c$$

$$13-\int \sec^2(x)dx = \tan(x) + c$$

$$14-\int \csc^2(x)dx = -\cot(x) + c$$

$$15-\int \sec(x)\tan(x)\, dx = \sec(x) + c$$

$$16-\int \csc(x)\cot(x)dx = -\csc(x) + c$$

$$17-\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c \quad , \quad \int \frac{-dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\frac{x}{a} + c$$

$$18-\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c \quad , \quad \int \frac{-dx}{a^2 + x^2} = \frac{1}{a}\cot^{-1}\frac{x}{a} + c$$

$$19-\int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}+c \quad , \qquad \int \frac{-dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\csc^{-1}\frac{x}{a}+c$$

Example:- Evaluate

$$1-\int (x^4+x^{-3})\,dx=\frac{x^5}{5}-\frac{x^{-2}}{2}+c$$

$$2-\int \frac{x^5-x^7}{x^2}\,dx=\int (x^3-x^5)\,dx=\frac{x^4}{4}-\frac{x^6}{6}+c$$

$$3-\int \frac{dx}{x+2}\,dx=\ln|x+2|+c$$

$$4-\int e^{2x-10}\,dx=\frac{1}{2}e^{2x-10}+c$$

$$5-\int 3^{x-5}\,dx=\frac{3^{x-5}}{\ln 3}+c$$

$$6-\int xe^{x^2}\,dx=\frac{1}{2}e^{x^2}+c$$

$$7-\int (x+1)^3\,dx=\frac{(x+1)^4}{4}+c$$

$$8-\int \sec^2(x+1)\,dx=\tan(x+1)+c$$

$$9-\int (x^3+2)^2\,dx \qquad \text{H.W.}$$

$$10-\int x\sqrt{1-x^2}\,dx \qquad \text{H.W.}$$

$$11-\int \frac{e^x}{e^x-1}\,dx \qquad \text{H.W.}$$

$$12-\left(2\int \frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}\,dx\right) \qquad \text{H.W.}$$

Lecture Ten

2- Integration by Partial fractions

To find the Integration of a function $F(x)=\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are Polynomials. If the degree of $f(x)$ is less than the degree of $g(x)$, then we need to factories $g(x)$ into linear factors.

Example:- Evaluate the following integrals

$$1-\int \frac{5x+5}{x^2+3x-4}\,dx$$

Solve:

$$\int \frac{5x+5}{x^2+3x-4}\,dx=\int \frac{5x+5}{(x-1)(x+4)}\,dx$$

$$=\int \frac{A}{(x-1)}\,dx+\int \frac{B}{(x+4)}\,dx$$

Now, we will find A and B

$$\frac{5x+5}{x^2+3x-4}=\frac{5x+5}{(x-1)(x+4)}$$

$$=\frac{A}{(x-1)}+\frac{B}{(x+4)}$$

$$= \frac{A(x+4)+B(x-1)}{(x-1)(x+4)}$$

See that

$$5x+5 = A(x+4)+B(x-1)$$

So to find A let

$$x=1 \Rightarrow 5(1)+5 = A(1+4) \Rightarrow 10 = 5A \Rightarrow A = 2$$

Also to find B let

$$x=-4 \Rightarrow 5(-4)+5 = B(-4-1) \Rightarrow -15 = -5B \Rightarrow B = 3$$

$$\int \frac{5x+5}{x^2+3x-4}dx = \int \frac{A}{(x-1)}dx + \int \frac{B}{(x+4)}dx$$

$$= \int \frac{2}{(x-1)}dx + \int \frac{3}{(x+4)}dx$$

$$= 2\ln|x-1| + 3\ln|x+4| + c$$

$$2 - \int \frac{1}{x(x^2+1)^2}dx$$

Solve:

$$\int \frac{dx}{x(x^2+1)^2} = \int \frac{A}{x}dx + \int \frac{Bx+C}{x^2+1}dx + \int \frac{Dx+E}{(x^2+1)^2}dx$$

Now, we will find A, B, C, D and E

$$\frac{1}{x(x^2+1)^2} = \frac{A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)x}{x(x^2+1)^2}$$

$$1 = A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)x$$

$$1 = A(x^4+2x^2+1) + B(x^4+x^2)C(x^3+x) + (Dx^2+Ex)$$

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

If we equate coefficients, we get

$$A+B=0 \;,\; C=0 \;, 2A+B+D=0 \;,\; C+E=0 \;,\; A=1$$

Then

$$A=1 \;,\; B=-1 \;,\; C=0 \;,\; D=-1 \;\; and \;\; E=0$$

$$\int \frac{dx}{x(x^2+1)^2} = \int \frac{A}{x}dx + \int \frac{Bx+C}{x^2+1}dx + \int \frac{Dx+E}{(x^2+1)^2}dx$$

$$= \int \frac{1}{x}dx + \int \frac{-x}{x^2+1}dx + \int \frac{-x}{(x^2+1)^2}dx$$

$$= \int \frac{1}{x}dx - \int \frac{x}{x^2+1}dx - \int \frac{x}{(x^2+1)^2}dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C$$

$$= \ln|x| - \ln\sqrt{x^2+1} + \frac{1}{2(x^2+1)} + C$$

$$= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + C$$

$$3-\int \frac{1}{x^2-1}\,dx \qquad \text{H.W.}$$

Lecture Eleven

3- Integration by Parts

The formula

$$\int u\,dv = uv - \int v\,du$$

Consider

$$w = u\cdot v \Longrightarrow dw = u\cdot dv + v\cdot du \Longrightarrow u\cdot dv = dw - vdu$$

$$\int u\,dv = \int dw - \int v\,du = w - \int v\,du$$

$$\int u\,dv = u\cdot v - \int v\,du \qquad \text{where} \quad w = u\cdot v$$

Example:- Evaluate the following integrals

$$1-\int \ln x\,dx$$

Solve:

$$\int \ln x\,dx = u\cdot v - \int v\,du$$

$$u = \ln x \qquad \Longrightarrow \qquad du = \frac{dx}{x}$$

$$dv = dx \qquad \Longrightarrow \qquad v = x$$

$$\int \ln x\,dx = x\ln x - \int x\frac{1}{x}dx$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + c$$

$$2-\int \tan^{-1} x\,dx$$

Solve:

$$\int \tan^{-1} x\,dx = u\cdot v-\int v\,du$$

$$u=\tan^{-1} x \quad \Longrightarrow \quad du=\frac{dx}{1+x^2}$$

$$dv=dx \quad \Longrightarrow \quad v=x$$

$$\int \tan^{-1} x\,dx = x\tan^{-1} x-\int \frac{x\,dx}{1+x^2}$$

$$= x\tan^{-1} x-\frac{1}{2}\ln(1+x^2)+c$$

$$3-\int e^x \sin x\,dx$$

Solve:

$$\int e^x \sin x\,dx = u\cdot v-\int v\,du$$

$$u=e^x \quad \Longrightarrow \quad du=e^x dx$$

$$dv=\sin x\,dx \quad \Longrightarrow \quad v=-\cos x$$

$$\int e^x \sin x\,dx = -e^x\cos x+\int e^x \cos x\,dx$$

And

$$\int e^x \cos x\,dx = u\cdot v-\int v\,du$$

$$u=e^x \quad \Longrightarrow \quad du=e^x dx$$

$$dv=\cos x\,dx \quad \Longrightarrow \quad v=\sin x$$

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$$

Then

$$\int e^x \sin x\, dx \;= -e^x \cos x + \int e^x \cos x\, dx$$

$$\int e^x \sin x\, dx \;= -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

$$2\int e^x \sin x\, dx \;= -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x\, dx \;= \frac{1}{2} e^x (\sin x - \cos x)$$

$4 - \int x e^x\, dx$ $\qquad\qquad$ $\mathbf{H.W.}$

4- Definite Integrals

The quantity

$$\int_a^b f(x)\, dx$$

Is called the Definite Integral of $f(x)$ from a to b. The numbers a and b are known as the lower and upper limits of the integral.

To see how to evaluate a definite integral consider the following example.

Example:- Find

$$1-\int_{1}^{4} x^2 dx$$

Solve:

$$\int_{1}^{4} x^2 dx = \left[\frac{x^3}{3}+c\right]_1^4$$

$$\left[\frac{x^3}{3}+c\right]_1^4 = (\text{evaluate at upper limit}) - (\textit{evaluate at lower limit})$$

$$\left[\frac{x^3}{3}+c\right]_1^4 = \left(\frac{(4)^3}{3}+c\right)-\left(\frac{(1)^3}{3}+c\right)$$

$$=\frac{64}{3}+c-\frac{1}{3}-c$$

$$=\frac{64}{3}+\frac{1}{3}=21$$

$$2-\int_{0}^{\frac{\pi}{2}} \cos x\, dx$$

Solve:

$$\int_{0}^{\frac{\pi}{2}} \cos x\, dx = \Big[\sin x\Big]_0^{\frac{\pi}{2}}$$

$$\Big[\sin x\Big]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right)-\sin(0)=1-0=1$$

$$3-\int_{0}^{1} x^2 dx$$

Solve:

$$\int_{0}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

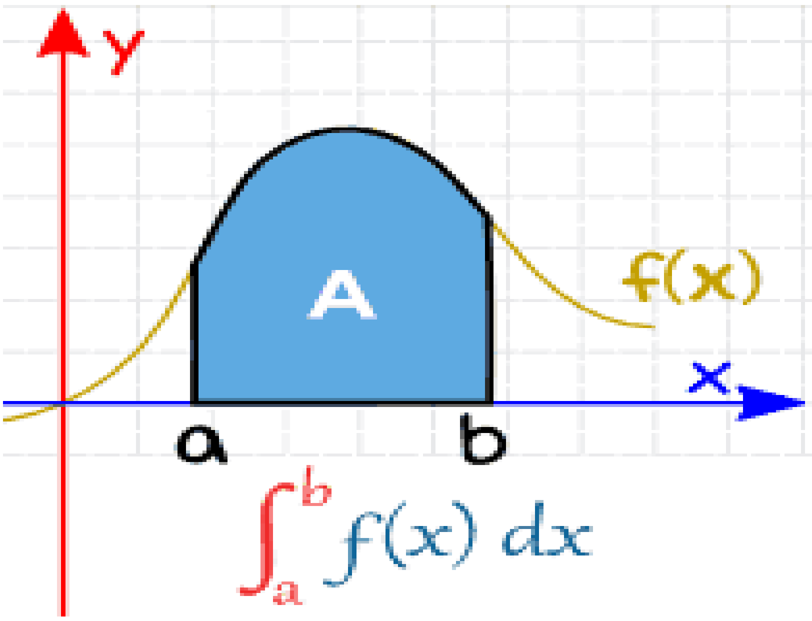
$$4-\int_{0}^{1} e^{2x} dx$$

Lecture Twelve

5- Some Properties of Definite Integral

If $a \leq x \leq b$, then

$$\int_{a}^{b} f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$



$$1-\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$2-\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \text{where} \quad c \in [a,b]$$

$$3-\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

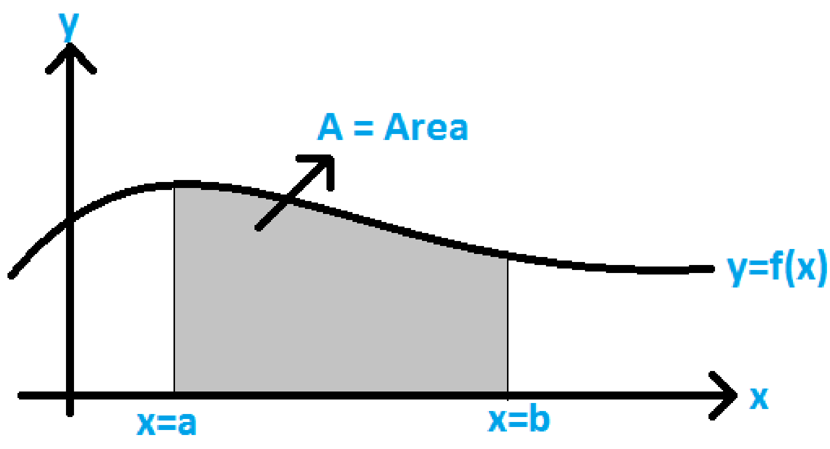
$$4-\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

6- Application on Integral

1- Area under the Graph

We have the law

$$A = \int_a^b f(x)dx$$



Example:- Find the Area bounded by $y = x^3$ from $x = -2$ to $x = 2$

Solve:

$$A = \int_0^2 f(x)dx + \left|\int_{-2}^0 f(x)dx\right|$$

$$= \int_0^2 x^3 dx + \left|\int_{-2}^0 x^3 dx\right|$$

$$= \left[\frac{x^4}{4}\right]_0^2 + \left|\left[\frac{x^4}{4}\right]_{-2}^0\right|$$

$$= 4 + 4 = 2$$

Example:- Find the Area bounded by $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

Solve:

$$A = \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= \left[\sin x\right]_0^{\frac{\pi}{2}}$$

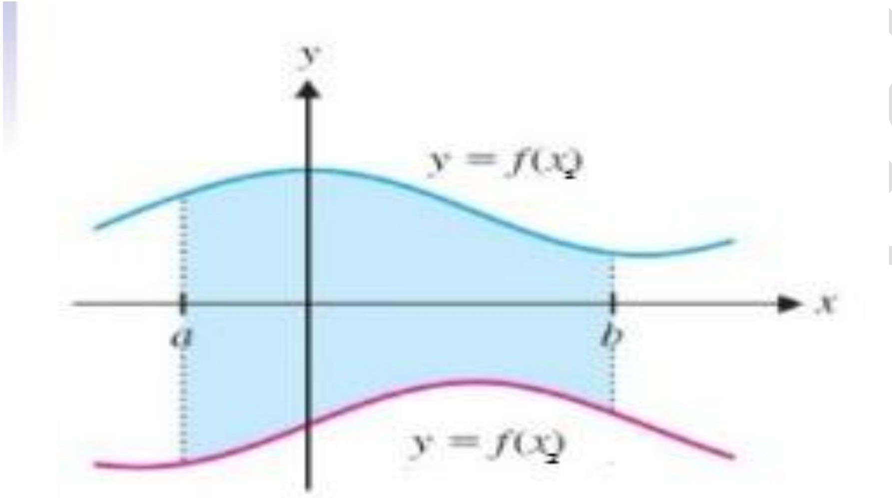
$$= \sin\frac{\pi}{2} - \sin 0 = 1$$

Example:- Find the Area bounded by $y = \frac{1}{2}x^2$ from $x = 1$ to $x = 3$ H.W.

2- Area between tow carvers

We have the law

$$A = \int_{a}^{b} [f(x_1) - f(x_2)]dx$$



Example:- Find the Area bounded between the carve $y = x^2$ and the line $y = x + 2$

Solve:

$$x + 2 = x^2 \Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x - 2)(x + 1) = 0$$

$$\Rightarrow x = 2 \quad or \quad x = -1$$

We have the tow points $(2.4), (-1, 1)$

$$A = \int_{-1}^{2} [(x + 2) - x^2]dx = \int_{-1}^{2} (x + 2 - x^2)dx$$

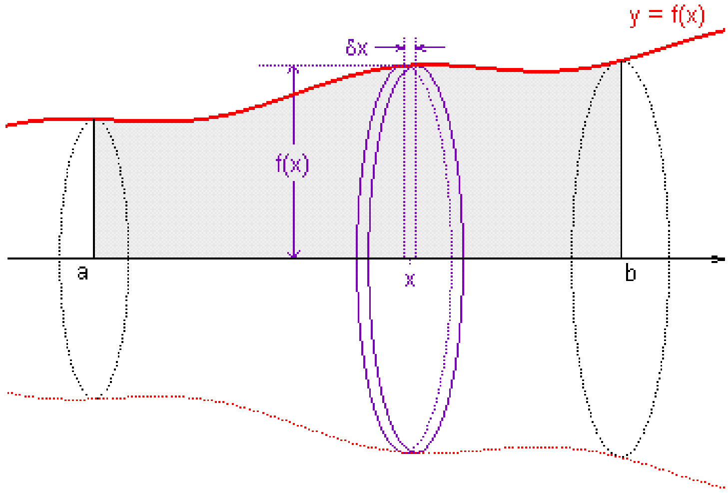
$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \left(\frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}\right) - \left(\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right)$$

$$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{10}{3} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2}$$

3- Volumes

We have law

$$V = \int_a^b A(x)dx = \int_a^b \pi y^2 dx$$



Example:- Find the Volume generated by rotating the bounded area by $y = x^2$ and the line $x = 4$ about x-axis

Solve:

$$V = \int_a^b A(x)dx = \int_a^b \pi y^2 dx$$

$$= \int_0^4 \pi x^4 dx = \pi \int_0^4 x^4 dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^4 = \pi \frac{(4)^5}{5} = \pi \frac{1024}{5}$$